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## SOLUTION BY OTTO J. RAMLER, Catholic University of America.

We have the known definite integral

$$\int_{-\infty}^{\infty} e^{-z^2}dz = \sqrt{\pi}.$$

Now let $z = (x - a/x)$. Then

$$dz = (1 + a/x^2)dx,$$

and the integral becomes

$$\int_0^{\infty} e^{-(x-a/x)^2}(1 + a/x^2)dx = \sqrt{\pi},$$

or

$$\int_0^{\infty} e^{-(x-a/x)^2}dx + a\int_0^{\infty} e^{-(x-a/x)^2}\frac{dx}{x^2} = \sqrt{\pi}.$$

In the second integral, let $x = a/y$. Then $dx = -a/y^2dy$, and the second integral becomes

$$\int_0^{\infty} e^{-(a/y-y)^2}dy \equiv \int_0^{\infty} e^{-(y-a/y)^2}dy.$$

Hence,

$$\int_0^{\infty} e^{-(x-a/x)^2}dx + \int_0^{\infty} e^{-(y-a/y)^2}dy = \sqrt{\pi}$$

or

$$2\int_0^{\infty} e^{-(x-a/x)^2}dx = \sqrt{\pi}$$

or

$$2\int_0^{\infty} e^{-x^2-a^2/x^2+2a}dx = 2e^{2a}\int_0^{\infty} e^{-x^2-a^2/x^2}dx = \sqrt{\pi}.$$

Whence,

$$\int_0^{\infty} e^{-(x^2-a^2/x^2)}dx = \frac{\sqrt{\pi}}{2e^{2a}}.$$

Solved similarly by O. S. ADAMS.

**339 (Mechanics). Proposed by C. N. SCHMALL, New York City.**

A roll of cloth of very small uniform thickness $a$ is coiled up tightly in the form of a circular cylinder of diameter $d$ and is laid horizontally across a perfectly rough incline so that its axis is parallel to the intersection of the plane with the horizontal. It is then allowed to unroll (without slipping) down the plane. Neglecting the motion of its center of gravity in the direction perpendicular to the plane, show that it will unroll entirely in the time

$$T = \frac{\pi}{4}\sqrt{\frac{6d^2}{ag\sin\phi}},$$

where $\phi$ is the inclination of the plane to the horizontal plane, and $g$ is the acceleration of gravity.

## SOLUTION BY PAUL CAPRON, U. S. Naval Academy.

After $t$ seconds, let the radius of the cylinder be $r$; let $s$ be the distance its center of gravity has moved parallel to the plane; $\theta$, the angle through which it has turned; $W = mg = 2\rho gr^2$ its weight; and let the reaction against it at its point of application with the plane be composed of $R$ lbs. perpendicular to the plane and $S$ lbs. upward along the plane. The moment of inertia is then $I = \rho r^4$. Since $a$ is very small, assume

$$r = \frac{d}{2} - \frac{a\theta}{2\pi}, \qquad rd\theta = ds.$$

Using the dot and double dot to indicate first and second derivatives with respect to the time, we have the equations for motion of the center of gravity along and perpendicular to the plane and of the cylinder about its geometric axis:

$$\frac{d}{dt}(m\dot{s}) = W\sin\phi - S, \qquad \frac{d}{dt}(m\dot{r}) = R - W\cos\phi, \qquad \text{and} \qquad \frac{d}{dt}(I\dot{\theta}) = rS.$$

The second of these is to be neglected. Eliminating $S$ from the first and last, we have

$$[A]\qquad \frac{1}{r}\frac{d}{dt}(I\dot{\theta}) + \frac{d}{dt}(m\dot{s}) - W \sin\phi = 0.$$

During the motion, $r$ varies from $d/2$ to 0, $\theta$ from 0 to $\pi d/a$, and $s$ from 0 to $\pi d^2/4a$.

(I) The proposed result can be obtained as follows: Assume

$$(1)\ \frac{d}{dt}(m\dot{s}) = m\frac{d\dot{s}}{dt}; \qquad (2)\ \frac{d}{dt}(I\dot{\theta}) = I\frac{d\dot{\theta}}{dt}; \qquad (3)\ \frac{d\dot{s}}{dt} = r\frac{d\dot{\theta}}{dt}$$

Then $[A]$ becomes

$$[B]\qquad r\ddot{r} + \frac{ag\sin\phi}{3\pi} = 0.$$

Integrating, we find, if $k = ag\sin\phi/3\pi$, $\log(d/2r) = p^2$,

$$T = \frac{d}{\sqrt{2k}}\int_0^\infty e^{-p^2}dp = \frac{\pi}{4}\sqrt{\frac{6d^2}{ag\sin\phi}} = \frac{0.3392d}{\sqrt{a\sin\phi}}$$

These assumptions seem unwarranted, however.

(II) If we let $x = 2r/d = 1 - (a\theta/\pi d)$, we have

$$ds = rd\theta = \frac{d}{2}x\cdot d\left[\frac{\pi d}{a}(1-x)\right] = -\frac{\pi d^2}{2a}xdx; \qquad s = \frac{\pi d^2}{4a}(1-x^2);$$

$$\dot{\theta} = -\frac{\pi d}{a}\dot{x}, \qquad \dot{r} = \frac{d}{2}\dot{x}, \qquad \dot{s} = r\dot{\theta} = -\frac{\pi d^2}{2a}x\dot{x}, \qquad m = \rho d^2x^2/2, \qquad I = \rho d^4x^4/16.$$

Thus $[A]$ becomes

$$[C]\qquad 3x\ddot{x} + 10\dot{x}^2 + 10k = 0, \qquad \text{where} \qquad k = 2ga\sin\phi/5\pi d^2.$$

If $\dot{x}^2 = y$, $3xdy + 20(y + k)dx = 0$. Whence, by integration,

$$x^{20/3}(y+k) = \text{Const.} = k,$$

since at the start

$$x = \frac{2r}{d} = 1, \qquad y = \left(-\frac{a\dot{\theta}}{\pi d}\right)^2 = 0.$$

$$[D]\qquad x^{20/3}\dot{x}^2 = k(1 - x^{20/3}) \qquad \text{or} \qquad \frac{-x^{10/3}\dot{x}}{\sqrt{1-x^{20/3}}} = \sqrt{k}.$$

$\dot{x} < 0$ throughout the motion; and $t$ varies from 0 to $T$ as $x$ varies from 1 to 0. Hence,

$$T = \frac{1}{\sqrt{k}}\int_0^1 \frac{x^{10/3}dx}{\sqrt{1-x^{20/3}}}.$$

Since

$$\int_0^1 x^m(1-x^n)^p dx = \frac{1}{n}\cdot\frac{\Gamma(p+1)\Gamma\left(\frac{m+1}{n}\right)}{\Gamma\left(p+1+\frac{m+1}{n}\right)}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \qquad \text{and} \qquad \Gamma\left(\frac{13}{20}\right) = \frac{20}{13}\Gamma\left(\frac{33}{20}\right)$$

we have

$$T = \frac{3\pi d}{26}\sqrt{\frac{10}{ag\sin\phi}}\frac{\Gamma(1.65)}{\Gamma(1.15)} = 0.195d/\sqrt{a\sin\phi} \qquad \text{(units feet and seconds).}$$

(III) Reverting to the original equations, we find

$$-\frac{k^2}{u^3} - \frac{h^2}{v^3}\frac{dv}{du} = 0, \qquad \frac{dv}{du} = \frac{v^3k^2}{u^3h^2}$$

$$S = \frac{1}{r}\frac{d}{dt}(I\dot{\theta}) = -\frac{\pi\rho d^4}{8ax}\frac{d}{dt}(x^4\dot{x}) = -\frac{\pi\rho d^4x^2}{8a}(x\ddot{x} + 4\dot{x}x^2).$$

From [$C$],

$$S = -\frac{\pi\rho d^4x^2}{12a}(\dot{x}^2 - 5k).$$

From [$D$],

$$S = \frac{\pi\rho d^4x^2}{12a}k(6 - x^{-20/3}) = \frac{\rho g \sin\phi}{30}d^2x^2(6 - x^{-20/3}).$$

When

$$x^{-20/3} = 6, \qquad x = x_0 = .76433, \qquad \dot{x}^2 = 5k = 2ga\sin\phi/\pi d^2, \qquad \dot{r}^2 = ga\sin\phi/2\pi,$$
$$\dot{\theta}^2 = 2\pi g\sin\phi/a, \qquad \dot{s}^2 = \pi g d^2\sin\phi x_0^2/2a = .29210\pi g d^2\sin\phi/a.$$

From this point on, the discussion in (II) calls for a negative value of $S$. Again,

$$R = W\cos\phi + \frac{d}{dt}(m\dot{r}) = \frac{\rho d^2}{2}\left[gx^2\cos\phi + \frac{d^2}{2}\frac{d}{dt}(x^2\cdot\dot{x})\right] = \frac{\rho d^2x}{4}[2gx\cos\phi + (x\ddot{x} + 2\dot{x}^2)d]$$

From [$C$] and [$D$],

$$R = \tfrac{1}{2}\rho d^2gx\left[x\cos\phi - \frac{2a\sin\phi}{5\pi d}(1 + \tfrac{2}{3}x^{-20/3})\right].$$

When $x = x_0$,

$$R = \tfrac{1}{2}\rho d^2gx_0\left[x_0\cos\phi - \frac{2a\sin\phi}{\pi d}\right];$$

and if

$$\tan\phi \ngtr \frac{\pi d x_0}{2a}, \qquad R \nless 0.$$

In general,

$$R > 0 \text{ as long as } \tan\phi < \frac{5\pi d}{2a}\cdot\frac{x}{1 + \frac{2}{3}x^{-20/3}}.$$

As $a$ is very small, there is no point in considering the condition $R > 0$ as a further restriction. As soon as $S$ becomes zero

$$\left(\text{after } T_0 = \frac{1}{\sqrt{k}}\int_{x_0}^{1}\frac{x^{10/3}dx}{\sqrt{1 - x^{20/3}}}\text{ secs.}\right),$$

the cylinder begins to rotate so rapidly that the uncoiled cloth is no longer laid down and the cylinder ceases to resemble even remotely a rigid body. It would seem, therefore, that without additional data, the problem is not to be solved.

IV. It may be a reasonable assumption that after $T_0$ secs., $S$ continues to be zero, the roll loosening just enough to effect this condition, and that the cloth is still laid down flat, so that $\pi dm = -2\rho a ds$. [$\rho = \pi lw/2g$, if $l$ is the length of the cylinder in feet, and $w$ its weight in lbs. per cu. ft.]

If this is the case, $m = \frac{1}{2}\rho d^2x_0^2 - (2/\pi)\rho as$, and the first of the equations of motion is

$$\frac{d}{dt}(m\dot{s}) = \left(\tfrac{1}{2}\rho d^2x_0^2 - \frac{2}{\pi}\rho as\right)\ddot{s} - \frac{2}{\pi}\rho a\dot{s}^2 = g\sin\phi\left(\tfrac{1}{2}\rho d^2x_0^2 - \frac{2}{\pi}\rho as\right),$$

or $(s - c)(\ddot{s} - g\sin\phi) + \dot{s}^2 = 0$. Letting $\dot{s}^2 = z$, $(c = \pi d^2x_0^2/4a)$,

$$(s - c)dz + [2z - 2g\sin\phi(s - c)]ds = 0.$$

In III, when $x = x_0$, $t = T_0$,

$$s = \frac{\pi d^2}{4a}(1 - x_0^2) = \frac{\pi d^2}{4a} - c, \qquad \dot{s}^2 = \frac{\pi g d^2x_0^2\sin\phi}{2a} = 2gc\sin\phi.$$

Taking these final conditions of the phase of the motion according to III as the initial conditions for the present discussion, we have

$$z = \dot{s}^2 = \tfrac{2}{3}g\sin\phi\left[(s - c) + \frac{4c^3}{(s - c)^2}\right].$$

The distance to be traversed is $(\pi d^2/4a) - ((\pi d^2/4a) - c) = c$. Hence, the duration of this second phase is

$$T_1 = \sqrt{\frac{3}{2g \sin \phi}} \int_0^c \frac{(s - c)ds}{\sqrt{4c^3 + (s - c)^3}} \sqrt[3]{2}\sqrt{c} \int_\alpha^0 \frac{xdx}{\sqrt{1 + x^3}}, \qquad \text{where} \qquad \alpha = -\tfrac{1}{2}\sqrt[3]{2}.$$

The full time of the motion is thus $T = T_0 + T_1$; or

$$T = \sqrt{\frac{5\pi d^2}{2ga \sin \phi}} \int_{x_0}^1 \frac{x^{10/3}dx}{\sqrt{1 - x^{20/3}}} + \sqrt{\frac{3\pi d^2 x_0^2}{8ga \sin \phi}} \sqrt[3]{2} \int_\alpha^0 \frac{xdx}{\sqrt{1 + x^2}}.$$

It is more convenient to write the first of these integrals as

$$\int_{x_0}^1 = \int_0^1 - \int_0^{x_0},$$

using the value found in (II).

By the Binomial Theorem,

$$\int_0^{x_0} \frac{x^{10/3}dx}{\sqrt{1 - x^{20/3}}} = \sum_0^\infty T_k$$

where now

$$T_0 = \tfrac{3}{13} x_0^{13/3}, \qquad T_1 = \tfrac{1}{22} x_0^{11}, \cdots \frac{T_{k+1}}{T_k} = \frac{20k + 13}{20k + 33} \; \frac{2k + 1}{12(k + 1)} x_0, \qquad (x_0 = 6^{-3/20}).$$

$$\Sigma T = \tfrac{3}{13} (.31204 + .010244 + .000797 + .000080 + .000009) = .07458,$$

$$\sqrt{\frac{5\pi}{2g}} \, \Sigma T = 0.03685,$$

$$\sqrt[3]{2} \int_{-\alpha}^0 \frac{xdx}{\sqrt{1 + x^3}} = \sum_0^\infty T_k \qquad \text{where now} \qquad T_0 = \tfrac{1}{4}, \cdots, \frac{T_{k+1}}{T_k} = \frac{3k + 2}{3k + 5} \cdot \frac{2k + 1}{8(k + 1)},$$

$$\Sigma T = .25 + .0125 + .001465 + .000222 + .000036 + .000007 + .000001 = .26423$$

$$\sqrt{\frac{3\pi x_0^2}{8g}} \, \Sigma T = 0.03865,$$

$$T = \frac{d}{\sqrt{a \sin \phi}} (0.1950 - 0.03685 + 0.03865) = \frac{0.1968d}{\sqrt{a \sin \phi}} \text{ (units feet and seconds).}$$

It may be noted that since $\pi dm = -2\rho a ds$ and $m = 2\rho r^2$, $2\pi r dr = -ads$, so that

$$r^2 = \frac{d^2 x_0^2}{4} - \frac{as}{\pi},$$

and when $s = c$, at the end of the motion,

$$r^2 = \frac{d^2 x_0^2}{4} - \frac{ac}{\pi} = 0.$$

The roll of cloth, intact would roll $\pi d^2/4a$ feet down the plane in

$$\sqrt{\frac{3\pi d^2}{4ga \sin \phi}} = \frac{0.2713d}{\sqrt{a \sin \phi}} \text{ secs.}$$

## II. Solution by William Hoover, Columbus, O.

Let $d = 2b$ (1), $r$ = the radius of the roll after any time $t$ from the beginning of motion, the width of the strip being taken as the unit, $x$ = the distance the roll has moved down the plane, and $\rho$, $k$, the density and radius of gyration about the axis. Then we have plainly

$$\rho\pi(b^2 - r^2) = \rho a x. \tag{2}$$

This is an instance in which we must employ the general definition of momentum and consider the mass, as well as the velocity, of the moving body as variables.

Let $F$ = the friction; then resolving along the inclined plane, for the linear motion we have

$$\frac{d}{dt}(\rho \cdot \pi r^2 \cdot \dot{x}) = \rho \cdot \pi r^2 g \sin \varphi - F \tag{3}$$

and for the angular motion, taking moments about the axis of the roll, $\theta$ being the angular motion corresponding to $x$, and so

$$x = r\theta \cdots \quad (4), \qquad \frac{d}{dt}(\rho\cdot\pi r^2\cdot k^2\dot{\theta}) = rF. \tag{5}$$

We have $k^2 = r^2/2$ (6), and from (4), $\dot{\theta} = \dot{x}/r$ (7). Substituting (6) and (7) in (5) and eliminating $F$ from the resulting equation and (3), developing the derivatives and reducing,

$$\ddot{x} + \frac{7}{3}\cdot\frac{\dot{r}}{r}\dot{x} = \frac{2g\sin\phi}{3}. \tag{8}$$

Now, from (2), $\dot{x} = -(2\pi r/a)\dot{r}$ (9), $\ddot{x} = -(2\pi/a)(\dot{r}^2 + r\ddot{r})$ (10), and substituting these in (8) and arranging,

$$\ddot{r} + \frac{10}{3r}\dot{r}^2 = -\frac{ag\sin\varphi}{3\pi r}. \tag{11}$$

Putting $\dot{r}^2 = y$ (12), (11) is thrown into the form $(dy/dr) + Py = Q$ (13), in which $P = 20/3r$ (14), and $Q = -(2ag\sin\varphi/3\pi r)$ (15), the integral (13) being

$$y = e^{-\int Pdr}\int e^{\int Pdr}Q + Ce^{-\int Pdr}. \tag{16}$$

Substituting (14) and (15) in (16) and performing the operations indicated,

$$y = -\frac{2ag\sin\varphi}{3\pi}r^{-20/3}\cdot\frac{3}{20}r^{20/3} + Cr^{-20/3} = -\frac{2ag\sin\varphi}{3\pi} + Cr^{-20/3}. \tag{17}$$

But when $r = b$, $y = 0$, and, hence, $C = agb^{20/3}\sin\varphi/(30\pi)$, and (17) becomes

$$\dot{r}^2 = \frac{dr^2}{dt^2} = \frac{ag\sin\varphi}{30\pi}\left(\frac{b^{20/3} - r^{20/3}}{r^{20/3}}\right), \tag{18}$$

giving for the required time

$$t = \sqrt{\frac{30\pi}{ag\sin\varphi}}\int_0^b \frac{r^{10/3}dr}{\sqrt{b^{20/3} - r^{20/3}}}. \tag{19}$$

The last factor may be put into the form,

$$I = \int_0^b \frac{r^{10/3}}{b^{10/3}}\left(1 + \frac{r^{20/3}}{2b^{20/3}} + \frac{3}{8}\frac{r^{40/3}}{b^{40/3}} + \cdots\right)dr, \tag{20}$$

the series appearing to converge suitably for the degree of approximation for $t$ in (19) by the conditions of the problem, the result though differing from that stated in the problem.

**347 (Mechanics). Proposed by E. B. ESCOTT, Kansas City, Mo.**

A cord $ABCD$ is suspended from points $A$ and $D$ which are 20 feet apart in horizontal distance. $D$ is 4 feet lower than $A$. At $B$ and $C$ are suspended weights 100 and 200 lbs. $AB = 8$ feet, $BC = 10$ feet, $CD = 12$ feet. Find angles $\alpha$, $\beta$, $\gamma$ made by $AB$, $BC$, $CD$, respectively, with the horizontal. Also find the tensions $T_1$, $T_2$, $T_3$ in $AB$, $BC$, $CD$.

## SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

For the equilibrium of $w_1 = 100$, resolving vertically and horizontally,

$$T_1\sin\alpha - T_2\sin\beta = w_1 = 100,$$

$$T_1\cos\alpha = T_2\cos\beta;$$

similarly for $w_2 = 200$,

$$T_3\sin\gamma + T_2\sin\beta = w_2 = 200,$$

$$T_3\cos\gamma = T_2\cos\beta.$$